Optimum Natural Convection in a Porous Medium Between a Vertical Polygonal Duct and a Heated Core

C. Y. Wang
Departments of Mathematics and
Mechanical Engineering,
Michigan State University,
East Lansing, MI 48824
e-mail: cywang@mth.msu.edu

The natural or free convection in a polygonal duct with a heated core is solved by eigenfunction expansion and boundary collocation. The optimal sizes of the core for maximum flow or maximum energy transport are determined. [DOI: 10.1115/1.4006101]

Keywords: free convection, Darcy–Brinkman, polygonal, core

1 Introduction

Natural convection in vertical ducts filled with a porous medium is important in a thermal siphon, thermal insulation, and nuclear waste disposal [1,2]. The appropriate governing equation for fluid flow is the Darcy–Brinkman equation [1]. The natural convection in a porous medium has been studied for vertical parallel plates [3–6] and concentric, annular ducts [7–11].

The purpose of the present paper is to study the fully developed natural convection in a vertical polygonal duct with a heated, circular core. Polygonal cross sections such as the square and hexagonal shapes are useful in stacking or space utilization. Particular emphasis is on determination of the optimum size of the core for maximal flow or energy transfer.

2 Formulation

Figure 1(*a*) shows a vertical, open ended, regular polygonal duct with an interior heated circular core. The space between the walls is filled with a porous medium. Let a prime denote a dimensional variable. The steady equilibrium temperature equation is

$$\nabla'^2 T' = 0 \tag{1}$$

The Darcy–Brinkman equation under the Boussinesq approximation is

$$\tilde{\mu}\nabla'^2 w' - \frac{\mu}{K} w' + g\tilde{\beta}(T' - T_e) = 0 \tag{2}$$

In the fully developed state, there is no pressure gradient and all variables are independent of the axial distance. Let the minimum half width of the duct be L and the radius of the core be bL. Normalize all lengths by L. If the core temperature is held constant at T_1, normalize the velocity and temperature by

$$w = \frac{w'}{g\tilde{\beta}L^2(T_1 - T_e)/\tilde{\mu}}, \quad \tau = \frac{T' - T_e}{T_1 - T_e} \tag{3}$$

Let (r, θ) be normalized cylindrical coordinates with the origin at the symmetry axis. Equations (1) and (2) become

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\tau = 0,$$
$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)w - s^2 w + \tau = 0 \tag{4}$$

Here s is an important nondimensional parameter characterizing the porous medium. The clear fluid case is recovered when $s = 0$. Due to symmetry, we need only to consider the repeated region shown in Fig. 1(*b*), where all lengths have been normalized by L. If the polygon has M sides, the opening angle is $\beta = \pi/M$ and the region is given by $0 \le \theta \le \beta,\ b \le r \le \sec\theta$. The boundary conditions are that τ is unity on the core, and is zero on the outer boundary, and that w is zero on all walls. After the temperature and velocity profiles are found, the flow rate Q and the energy transport E can be integrated

$$Q = \iint w r dr d\theta, \quad E = \iint w\tau r dr d\theta \tag{5}$$

Q and E represent the maximum flow and energy transport through natural convection, regardless of entrance effects.

In the case, the boundary condition on the core is constant flux q instead of constant temperature, normalize by

$$w = \frac{w'}{g\tilde{\beta}qL^3/(k\tilde{\mu})}, \quad \tau = \frac{T' - T_e}{qL/k} \tag{6}$$

Then Eqs. (4) remain the same, but the temperature boundary condition on the core is

$$\frac{\partial \tau}{\partial r}(b, \theta) = -1 \tag{7}$$

3 The Annular Duct

The annular duct serves as a limiting case for our polygonal duct with a core. Since the problem is axisymmetric, it has exact solutions in modified Bessel functions. For the constant temperature case, the solutions to Eqs. (4) are

$$\tau = \frac{\ln r}{\ln b} \tag{8}$$

$$w = \frac{1}{s^2}\left[\frac{\ln r}{\ln b} - \frac{K_0(s)I_0(sr) - I_0(s)K_0(sr)}{K_0(s)I_0(sb) - I_0(s)K_0(sb)}\right] \tag{9}$$

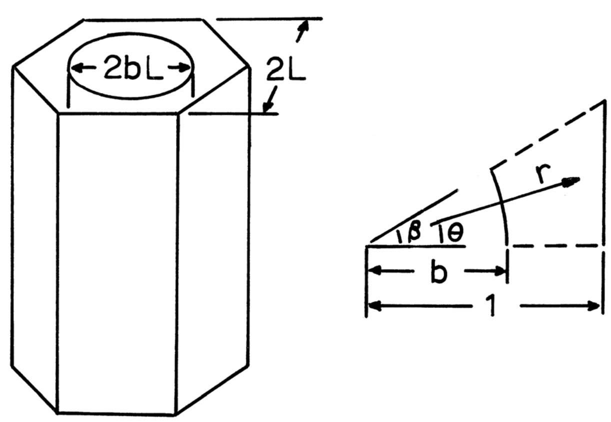

Fig. 1 (*a*) The polygonal duct with a heated core (*b*) the domain of computation

Contributed by the Heat Transfer Division of ASME for publication in the JOURNAL OF HEAT TRANSFER. Manuscript received July 27, 2011; final manuscript received November 9, 2011; published online May 29, 2012. Assoc. Editor: Andrey Kuznetsov.

Table 1 Optimum core radii for the annular duct, constant temperature case. Maximum Q and E are in parentheses.

s	$b\,(Q)$	$b\,(E)$
0	0.0948 (0.05697)	0.1903 (0.01784)
1	0.1055 (0.05217)	0.1997 (0.01674)
2	0.1316 (0.04202)	0.2224 (0.01424)
5	0.2186 (0.01899)	0.2960 (0.00750)
10	0.2988 (0.00687)	0.3644 (0.00310)
20	0.3521 (0.00203)	0.4157 (0.00101)

When $s \to 0$

$$w \sim \frac{(1-b^2)\ln r - \ln b\,[1 - r^2 + (r^2 - b^2)\ln r]}{4(\ln b)^2} \tag{10}$$

which agrees with Ref. [12] for the clear fluid case. When $b \to 0$

$$w \sim \frac{K_0(sr) + \ln r - K_0(s)I_0(sr)/I_0(s)}{s^2 \ln b} \tag{11}$$

Notice both τ and w are proportional to $1/|\ln b|$ for small b. Using Eq. (5) the flow rate is

$$Q = \frac{\pi}{s^2}\left\{\frac{1-b^2}{2|\ln b|} - \frac{2 + b^2 s^2[I_2(bs)K_0(s) - I_0(s)K_2(bs)]}{s^2[I_0(bs)K_0(s) - I_o(s)K_0(bs)]}\right\} \tag{12}$$

Unfortunately, there is no closed form integral for the energy E, which is integrated numerically. As the core radius b is increased, both Q and E rise singularly from zero, reaches a maximum, and then decreases to zero as b approaches one. Of interest is the optimum core radius for maximum flow or energy transfer given in Table 1.

If the core is at constant flux, the solutions are

$$\tau = -b\ln r \tag{13}$$

$$w = \frac{1}{s^2}\left[-b\ln r + b\ln b\,\frac{K_0(s)I_0(sr) - I_0(s)K_0(sr)}{K_0(s)I_0(sb) - I_0(s)K_0(sb)}\right] \tag{14}$$

By taking the asymptotic limit as $s \to 0$, Eq. (14) gives the clear fluid solution

$$w \sim \frac{b}{4\ln b}\{\ln b[1 - r^2 + (r^2 - b^2)\ln r] - (1-b)\ln r\} \tag{15}$$

For the constant flux case, both temperature and velocity are regular for small b. The flow rate is

$$Q = \frac{b\pi}{s^2}\left\{\frac{1-b^2}{2} + \frac{\ln b\{2 + b^2 s^2[I_2(bs)K_0(s) - I_0(s)K_2(bs)]\}}{s^2[I_0(bs)K_0(s) - I_o(s)K_0(bs)]}\right\} \tag{16}$$

The energy transfer is similarly numerically integrated. The optimum core sizes are given in Table 2.

Table 2 Optimum core radii for the annular duct, constant flux case. Maximum Q and E are in parentheses.

s	$b\,(Q)$	$b\,(E)$
0	0.2095 (0.01667)	0.2841 (0.002123)
1	0.2160 (0.01563)	0.2884 (0.002021)
2	0.2318 (0.01323)	0.2991 (0.001773)
5	0.2849 (0.00663)	0.3350 (0.000996)
10	0.3320 (0.00250)	0.3664 (0.000419)
20	0.3602 (0.000745)	0.3865 (0.000136)

4 The Polygonal Duct

Exact solutions do not exist for polygonal ducts and numerical means are needed for solving the temperature distribution and the velocity profile. We shall use the method of eigenfunction expansion and boundary collocation for a polygonal duct of M sides. Figure 1(*b*) shows the domain of computation. For the constant temperature case, let

$$\tau(r,\theta) = \frac{\ln r}{\ln b} + A_0 \ln\left(\frac{r}{b}\right) + \sum_{n=1}^{N-1} A_n \cos(\alpha\theta)\left[\left(\frac{r}{b}\right)^{\alpha} - \left(\frac{b}{r}\right)^{\alpha}\right] \tag{17}$$

where $\alpha = Mn$. Equation (17) satisfies the boundary conditions at the core $r = b$ and the reflective conditions at $\theta = 0, \beta$. At the straight boundary on the polygon, we set $\tau = 0$ at N equally spaced points, yielding algebraic equations which can be inverted for A_n.

For the porous media case, the general solution which satisfies Eq. (4), the boundary conditions at $r = b$ and the reflective conditions at $\theta = 0, \beta$, is

$$w(r,\theta) = \frac{1}{s^2}\left\{\tau(r,\theta) - \frac{I_0(sr)}{I_0(sb)} + B_0 Z_0(r) + \sum_{n=1}^{N-1} B_n \cos(\alpha\theta) Z_n(r)\right\} \tag{18}$$

On the outer wall w is set to zero at N points. The coefficients B_n are inverted similarly. Convergence is fairly fast, since a four-digit accuracy is attained for keeping eight terms in the series.

For the constant flux case, Eqs. (17) and (18) are replaced by

$$\tau = -b\ln r + A_0 + \sum_{n=1}^{N-1} A_n \cos(\alpha\theta)\left[\left(\frac{r}{b}\right)^{\alpha} + \left(\frac{b}{r}\right)^{\alpha}\right] \tag{19}$$

$$w = \frac{1}{s^2}\left[\tau + (b\ln b - A_0)\frac{I_0(sr)}{I_0(sb)} - \sum_{n=1}^{N-1} 2A_n \cos(\alpha\theta)\frac{I_\alpha(sr)}{I_\alpha(sb)} + B_0 Z_0(r) + \sum_{n=1}^{N-1} B_n \cos(\alpha\theta) Z_n(r)\right] \tag{20}$$

The boundary conditions at the outer boundary are similarly matched at N discrete points.

For clear fluid and the constant temperature case, we set $s = 0$ in Eq. (4) and use Eq. (17) to find the velocity, which is different from Eq. (18).

$$w = \frac{(r^2 - b^2)}{4\ln b}(1 - \ln r) - \frac{(r^2 - b^2)}{4}A_0 + \sum_{n=1}^{N-1}\frac{A_n}{4}\cos(\alpha\theta)\left[\frac{r^{2-\alpha}b^{\alpha}}{(1-\alpha)} - \frac{r^{2+\alpha}}{b^{\alpha}(1+\alpha)} - \frac{2b^2\alpha}{1-\alpha^2}\left(\frac{r}{b}\right)^{\alpha}\right] + B_0\ln\left(\frac{r}{b}\right) + \sum_{n=1}^{N-1} B_n\cos(\alpha\theta)\left[\left(\frac{r}{b}\right)^{\alpha} - \left(\frac{b}{r}\right)^{\alpha}\right] \tag{21}$$

For the constant flux case Eqs. (4) and (19) give

$$w = \frac{b(r^2 - b^2)}{4}(\ln r - 1) - \frac{(r^2 - b^2)}{4}A_0 - \sum_{n=1}^{N-1}\frac{A_n}{4}\cos(\alpha\theta)\left[\frac{r^{2-\alpha}b^{\alpha}}{(1-\alpha)} + \frac{r^{2+\alpha}}{b^{\alpha}(1+\alpha)} - \frac{2b^2}{1-\alpha^2}\left(\frac{r}{b}\right)^{\alpha}\right] + B_0\ln\left(\frac{r}{b}\right) + \sum_{n=1}^{N-1} B_n\cos(\alpha\theta)\left[\left(\frac{r}{b}\right)^{\alpha} - \left(\frac{b}{r}\right)^{\alpha}\right] \tag{22}$$

Similar to the annular duct, polygonal ducts also have optimum core sizes. Figure 2 shows the effect of the decrease in the number

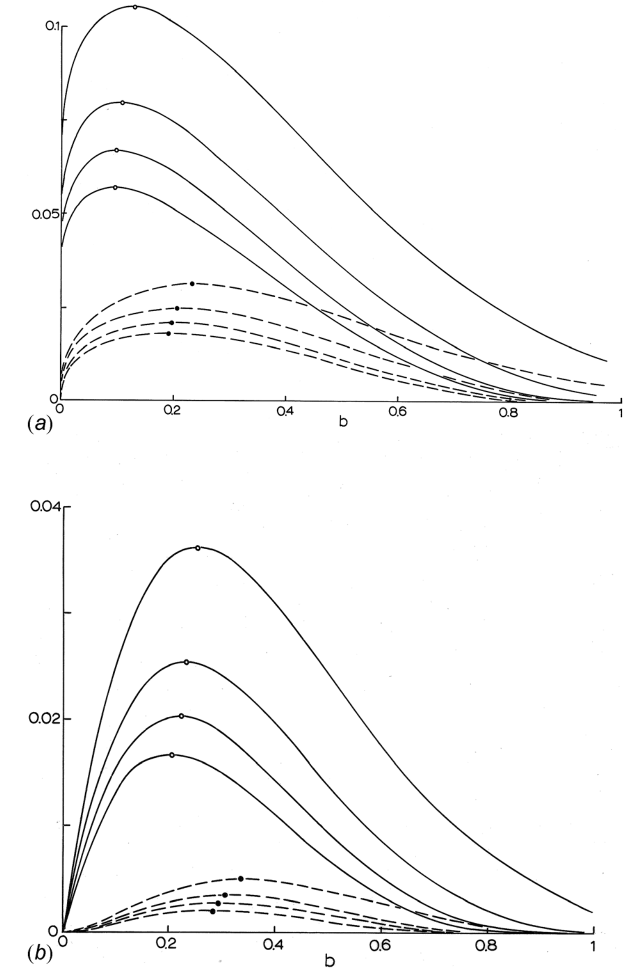

Fig. 2 (*a*) Effect of number of sides, constant temperature case, $s = 0$, (*b*) constant flux case, $s = 0$. Solid curves are for Q and dashed curves are for E. Each set from top: $M = 3, 4, 6, \infty$. The small dots are maxima locations.

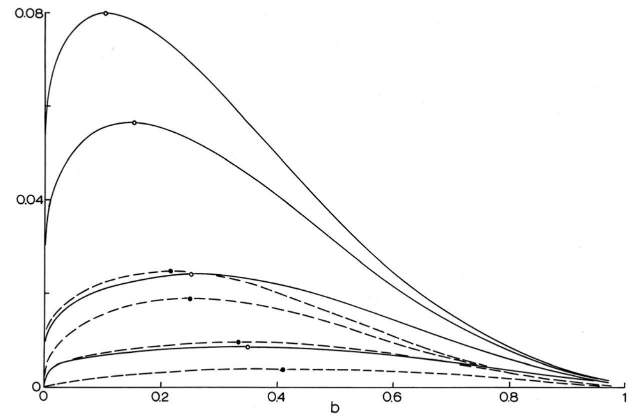

Fig. 3 Effect of the porous parameter, constant temperature case, $M = 4$. Solid curves are for Q and dashed curves are for E. Each set from top: $s = 0$, 2, 5, 10. The small dots are maxima locations.

of sides M increases flow and energy transport. The optimum core size also changed. The increase in the porous parameter s suppresses flow and energy transport as shown in Fig. 3.

The optimum core radii for polygonal ducts are given in Table 3 for the constant temperature case and Table 4 for the constant flux case.

5 Discussions

The natural convection in a polygonal duct with a heated core is solved for the first time. Polygonal ducts include the triangular, square, and hexagonal ducts, all which are stackable. The length scale L (half width) is used due to stacking constraints. The problem depends heavily on the porous media parameter s, which ranges from zero for clear fluids to vary large for packed beds.

We assumed a fully developed state, which serves as a limit for flow and energy transport, regardless of entrance effects. For the length of entrance region, one may consult Refs. [13,14].

The temperature distribution is independent of s, but not the convective flow. Although the formulations are different for s

Table 3 Optimum core radii for the triangular ($M = 3$), square ($M = 4$), and hexagonal ($M = 6$) ducts, constant temperature case. Maximum Q and E are in parentheses.

	$M = 3$		$M = 4$		$M = 6$	
s	b (Q)	b (E)	b (Q)	b (E)	b (Q)	b (E)
0	0.123 (0.1051)	0.233 (0.0314)	0.106 (0.0798)	0.210 (0.0246)	0.0989 (0.0664)	0.197 (0.0207)
1	0.135 (0.0938)	0.243 (0.0289)	0.119 (0.0720)	0.221 (0.0228)	0.107 (0.0604)	0.206 (0.0194)
2	0.179 (0.0714)	0.279 (0.0235)	0.155 (0.0564)	0.251 (0.0190)	0.143 (0.0481)	0.235 (0.0163)
5	0.290 (0.0289)	0.372 (0.0113)	0.254 (0.0240)	0.334 (0.0095)	0.234 (0.0211)	0.311 (0.0084)
10	0.382 (0.0099)	0.449 (0.0045)	0.340 (0.0084)	0.406 (0.0038)	0.315 (0.0075)	0.382 (0.0034)
20	0.436 (0.0028)	0.505 (0.0014)	0.394 (0.0024)	0.460 (0.0012)	0.370 (0.0022)	0.435 (0.0011)

Table 4 Optimum core radii for the triangular ($M = 3$), square ($M = 4$), and hexagonal ($M = 6$) ducts, constant flux case. Maximum Q and E are in parentheses.

	$M = 3$		$M = 4$		$M = 6$	
s	b (Q)	b (E)	b (Q)	b (E)	b (Q)	b (E)
0	0.254 (0.0363)	0.233 (0.0314)	0.231 (0.0255)	0.210 (0.00246)	0.219 (0.0202)	0.295 (0.00266)
1	0.265 (0.0332)	0.342 (0.0161)	0.241 (0.0236)	0.314 (0.00324)	0.226 (0.0188)	0.299 (0.00252)
2	0.290 (0.0267)	0.356 (0.00388)	0.260 (0.0195)	0.328 (0.00278)	0.245 (0.0152)	0.311 (0.00219)
5	0.351 (0.0117)	0.401 (0.00195)	0.319 (0.0092)	0.369 (0.00148)	0.296 (0.0077)	0.351 (0.00120)
10	0.403 (0.0041)	0.431 (0.00076)	0.368 (0.0033)	0.402 (0.00060)	0.348 (0.00285)	0.382 (0.00050)
20	0.425 (0.0012)	0.452 (0.00024)	0.397 (0.00097)	0.421 (0.00019)	0.376 (0.00084)	0.403 (0.00016)

nonzero or s zero, the results are identical in the limit, say $s=0.01$. For large s, a velocity boundary layer exists on the core.

Both an increase in the porous parameter s, or an increase in the number of sides M decrease Q and E. However, the optimum core size increases with increased s but decreases with increased M. Our optimum size tables will be useful in the design of natural convection apparatus.

Nomenclature

$A_n =$ constant coefficients
$b =$ normalized core radius
$B_n =$ constant coefficients
$E =$ energy integral defined by Eq. (5)
$g =$ gravitational acceleration (m/s^2)
$i =$ integer
$I_n, K_n =$ modified Bessel functions
$k =$ effective thermal conductivity of porous matrix (W/mK)
$K =$ permeability (m^2)
$L =$ minimum half width of polygon (m)
$M =$ number of sides
$n =$ integer
$N =$ number of collocation points
$q =$ heat flux (W/m^2)
$Q =$ normalized flow rate Eq. (5)
$r =$ radial coordinate normalized by L
$s =$ nondimensional parameter $L/\sqrt{\tilde{\mu}K/\mu}$
$T =$ temperature (K)
$T_e =$ ambient temperature (K)
$w =$ normalized axial velocity
$y =$ normalized Cartesian coordinate
$Z_n = I_n(sr)/I_n(sb) - K_n(sr)/K_n(sb)$
$\alpha = Mn$
$\beta = \pi/M$
$\tilde{\beta} =$ thermal expansion coefficient (Ns^2/Km^4)
$\mu =$ fluid viscosity (N s/m^2)
$\tilde{\mu} =$ effective viscosity of the porous medium (N s/m^2)
$\tau =$ normalized temperature distribution
$\theta =$ angle coordinate
$' =$ dimensional quantity

References

[1] Nield, D. A., and Bejan, A., 2006, *Convection in Porous Media*, 3rd ed., Springer, New York.
[2] Prasad, V., Kulacki, F. A., and Kulkarni, A. V., 1986, "Free Convection in a Vertical, Porous Annulus With Constant Heat Flux on the Inner Wall-Experimental Results," Int. J. Heat Mass Transfer, **29**, pp. 713–723.
[3] Kou, H. S., and Lu, K. T., 1993, "Combined Boundary and Inertia Effects for Fully Developed Mixed Convection in a Vertical Channel Embedded in a Porous Media," Int. Commun. Heat Mass Transfer, **20**, pp. 333–345.
[4] Al-Nimr, M. A., and Alkam, M. K., 2000, "Basic Fluid Flow Problems in Porous Media," J. Porous Media **3**, pp. 45–59.
[5] Al-Nimr, M. A., and El-Shaarawi, M. A. I., 1995, "Analytical Solution for Transient Laminar Fully Developed Free Convection in Vertical Channels," Heat Mass Transfer, **30**, pp. 241–248.
[6] Al-Nimr, M. A., and Haddad, O. M., 1999, "Fully Developed Free Convection in Open-Ended Vertical Channels Partially Filled With Porous Material," J. Porous Media **2**, pp. 179–189.
[7] Parang, M., and Keyhani, M., 1987, "Boundary Effects in Laminar Mixed Convection Flow Through an Annular Porous Medium," ASME Trans. J. Heat Transfer, **109**, pp. 1039–1041.
[8] Hasnaoui, M., Vasseur, P., Bilgen, E., and Robillard, L., 1995, "Analytical and Numerical Study of Natural Convection Heat Transfer in a Vertical Porous Annulus," Chem. Eng. Commun., **136**, pp. 77–94.
[9] Al-Nimr, M. A., 1995, "Fully Developed Free Convection in Open Ended Vertical Concentric Porous Annuli," Int. J. Heat Mass Transfer, **38**, pp. 1–12.
[10] Du, J. H., and Wang, B. X., 1999, "Mixed Convection in Porous Media Between Vertical Concentric Cylinders," Heat Trans. Asian Res., **28**, pp. 95–101.
[11] Jha, B. K., 2005, "Free Convection Flow Through an Annular Porous Medium," Heat Mass Trans., **41**, pp. 675–679.
[12] Wang, C. Y., 2010, "Optimization of Natural Convection in Open Vertical Ducts With Heated Cores," J. Thermophys. Heat Transfer, **24**, pp. 669–672.
[13] Hadim, H. A., and Chen, G., 1994, "Non-Darcy Mixed Convection in a Vertical Porous Channel with Asymmetric Heating," J. Thermophys. Heat Transfer, **8**, pp. 805–808.
[14] Al-Nimr, M. A., and Alkam, M., 1997, "Unsteady Non-Darcian Forced Convection Analysis in an Annulus Partly Filled With Porous Material," ASME Trans. J. Heat Trans., **119**, pp. 799–804.